

## Convex Hulls of Lévy Processes

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### Abstract

Let  $X(t)$ ,  $t \geq 0$ , be a Lévy process in  $\mathbb{R}^d$  starting at the origin. We study the closed convex hull  $Z_s$  of  $\{X(t) : 0 \leq t \leq s\}$ . In particular, we provide conditions for the integrability of the intrinsic volumes of the random set  $Z_s$  and find explicit expressions for their means in the case of symmetric  $\alpha$ -stable Lévy processes. If the process is symmetric and each its one-dimensional projection is non-atomic, we establish that the origin a.s. belongs to the interior of  $Z_s$  for all  $s > 0$ . Limit theorems for the convex hull of Lévy processes with normal and stable limits are also obtained.

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A number of existing results concern convex hulls of stochastic processes, especially the Brownian motion and random walks, see [2, 8, 17]. In contrary, considerably less is known about convex hulls of general Lévy processes with the exception of some results for symmetric  $\alpha$ -stable Lévy processes in  $\mathbb{R}^d$  with  $\alpha \in (1, 2]$ , see [6].

First, recall some concepts from convex geometry. Denote by  $\mathcal{K}_d$  the family of *convex bodies* (non-empty compact convex sets) in  $\mathbb{R}^d$ . For  $K, L \in \mathcal{K}_d$ , denote by  $K + L = \{x + y : x \in K, y \in L\}$  their Minkowski sum. It is known that the volume  $V_d(K + tL)$  is a polynomial in  $t \geq 0$  of degree  $d$ , see [15, Th. 5.1.7]. The *mixed volumes*  $V(K[j], L[d - j])$ ,  $j = 0, \dots, d$ , appear as coefficients of this polynomial, so that

$$V_d(K + tL) = \sum_{j=0}^d \binom{d}{j} t^{d-j} V(K[j], L[d - j]), \quad t \geq 0. \quad (0.1)$$

The mixed volume is a function of  $d$  arguments, and  $K[j]$  stands for its  $j$  arguments, all being  $K$ . The *intrinsic volumes* of a convex body  $K$  are normalised mixed volumes

$$V_j(K) = \frac{\binom{d}{j}}{\kappa_{d-j}} V(K[j], B^d[d - j]), \quad j = 0, \dots, d, \quad (0.2)$$

where  $B^d$  denotes the centred  $d$ -dimensional unit ball and  $\kappa_{d-j}$  is the  $(d - j)$ -dimensional volume of  $B^{d-j}$ . In particular,  $V_d(K)$  is the volume (or the Lebesgue measure),  $V_{d-1}(K)$  is half the surface area,  $V_{d-2}(K)$  is proportional to the integrated mean curvature,  $V_1(K)$  is proportional to the mean width of  $K$ , and  $V_0(K) = 1$ , see [15, Sec. 4.2, 5.3]. The *Hausdorff metric* between convex bodies is defined by

$$\rho_H(K, L) = \inf\{r \geq 0 : K \subset L + rB^d, L \subset K + rB^d\}.$$

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The  $j$ -dimensional volume of the parallelepiped spanned by  $u_1, \dots, u_j \in \mathbb{R}^d$  is denoted by  $D_j(u_1, \dots, u_j)$ .

Let  $X(t)$ ,  $t \geq 0$ , be a Lévy process in  $\mathbb{R}^d$  starting at the origin. We are interested in

$$Z_s = \text{cl conv}\{X(t) : 0 \leq t \leq s\},$$

where  $\text{cl}(\cdot)$  denotes the closure and  $\text{conv}(\cdot)$  the convex hull. It is easy to see that  $Z_s$  is a random closed convex set, see [10]. We shortly denote  $Z = Z_1$ . In the case of a Brownian motion, the expected intrinsic volumes of  $Z$  are known, see [2]. The integrability of intrinsic volumes of  $Z$  for  $X$  being a symmetric  $\alpha$ -stable Lévy process with independent coordinates and the expected mean width of  $Z$  were obtained in [6].

We start by establishing the integrability of intrinsic volumes in relation to the properties of the Lévy measure of the general Lévy process and then find explicit expressions for their first moments of all intrinsic volumes for the case of symmetric  $\alpha$ -stable Lévy processes with  $\alpha \in (1, 2]$ . For this purpose, we generalise results on expected random determinants from [5, 16].

It is also shown that the origin a.s. belongs to the interior of the convex hull of symmetric Lévy processes if the scalar product  $\langle X(t), u \rangle$  is non-atomic for each  $u \neq 0$ . As a direct consequence, we prove that  $X(s)$  a.s. belongs to the interior of the convex hull of  $\{X(t) : 0 \leq t \leq s\}$ . We also consider expectations of the  $L_p$ -generalisations of mixed volumes and prove limit theorems for the scaled  $Z_t$  as  $t \rightarrow \infty$ .

## 1 Integrability of the intrinsic volumes

The integrability of  $V_j(Z_s)^p$  for some  $p > 0$ ,  $s \geq 0$ , and all  $j = 0, \dots, d$  is equivalent to  $EV_d(Z_s + B^d)^p < \infty$ . Indeed, the Steiner formula [15, Eq. (4.1)] yields that

$$V_d(Z_s + B^d) = \sum_{j=0}^d \kappa_{d-j} V_j(Z_s).$$

Thus, the existence of the  $p$ th moment in the left-hand side is equivalent to the existence of the  $p$ th moments of all (non-negative) summands in the right-hand side.

The Lévy measure of the process  $X(t)$ ,  $t \geq 0$ , is denoted by  $\nu$ . Denote

$$\beta_\nu = \sup \left\{ \beta > 0 : \int_{\|x\| > 1} \|x\|^\beta \nu(dx) < \infty \right\}.$$

**Theorem 1.1.** *If  $0 \leq p < \beta_\nu$ , then  $EV_j(Z_s)^p < \infty$  for all  $j = 0, \dots, d$  and all  $s \geq 0$ .*

*Proof.* The result is obvious if  $X(t)$ ,  $t \geq 0$ , is a deterministic process, so we exclude this case in the following. The main idea is to split the path of the Lévy process into several parts with integrable volumes of their convex hulls. The random variables  $T_0 = 0$  and

$$T_i = \inf\{t \geq T_{i-1} : X(t) \notin X(T_{i-1}) + B^d\}, \quad i \geq 1, \quad (1.1)$$

form an increasing sequence of stopping times with respect to the filtration generated by the process, see [1, Cor. 8]. Since  $X(t)$  has unbounded support for each non-trivial Lévy process and any  $t > 0$  [14, Th. 24.3], these stopping times are a.s. finite. The random variables  $\tilde{T}_i = T_i - T_{i-1}$ ,  $i \geq 1$ , are independent identically distributed. We set  $\tilde{T}_0 = 0$  and consider the renewal process

$$N_s = \max\{k \geq 0 : \tilde{T}_0 + \dots + \tilde{T}_k \leq s\}.$$

It is easy to see that  $EN_s^j < \infty$  for all  $j \geq 1$ . Let  $I_k$  be the segment in  $\mathbb{R}^d$  with end-points at the origin and  $X(\min(T_k, s)) - X(\min(T_{k-1}, s))$ ,  $k \geq 1$ . The segments  $I_k$ ,  $k \geq 1$ , are

independent. Denote by  $\|I_k\|$  the length of  $I_k$ . It is obvious that  $\|I_k\| \leq 2 \sup_{0 \leq t \leq s} \|X_t\|$ . If  $p < \beta_\nu$ , then  $\mathbf{E}\|I_k\|^p < \infty$  by [14, Thms. 25.3, 25.18] and [14, Prop. 25.4].

Observe that  $Z_s \subset B^d$  if  $N_s = 0$  and otherwise

$$Z_s \subseteq I_1 + \cdots + I_{N_s} + B^d.$$

Thus,

$$V_d(Z_s + B^d) \leq V_d(I_1 + \cdots + I_{N_s} + 2B^d).$$

The right-hand side equals the linear combination of the mixed volumes

$$\tilde{V}_j = V((I_1 + \cdots + I_{N_s})[j], B^d[d-j]), \quad j = 0, \dots, d,$$

of the sets  $I_1 + \cdots + I_{N_s}$  and  $B^d$ , see [15, Th. 5.1.7]. Therefore, it suffices to show that  $\mathbf{E}[\tilde{V}_j^p 1_{N_s \geq 1}] < \infty$  for all  $j = 0, \dots, d$ . It is obvious that  $\tilde{V}_0 = V_d(B^d)$  is integrable, so assume  $j \geq 1$ . For all  $k \leq N_s$  and  $N_s \geq 1$ , let  $[-\zeta_k, \zeta_k]$  be the scaled translate of  $I_k$  that is symmetric with respect to the origin and such that  $\|\zeta_k\| = 1$ . It is well known that the mixed volumes are translation invariant. In view of (0.2), the McMullen–Matheron–Weil formula [7, Eq. (1.4)] yields

$$\mathbf{E}[\tilde{V}_j^p 1_{N_s \geq 1}] = \sum_{n=1}^{\infty} \mathbf{E} \left[ \left( \frac{2^j \kappa_{d-j}}{j! \binom{d}{j}} \sum_{1 \leq k_1, \dots, k_j \leq n} \|I_{k_1}\| \cdots \|I_{k_j}\| 2^{-j} D_j(\zeta_{k_1}, \dots, \zeta_{k_j}) \right)^p 1_{N_s=n} \right].$$

Since  $D_j(\zeta_{k_1}, \dots, \zeta_{k_j}) \leq 1$ , we have

$$\begin{aligned} \mathbf{E}[\tilde{V}_j^p 1_{N_s \geq 1}] &\leq \left( \frac{(d-j)! \kappa_{d-j}}{d!} \right)^p \sum_{n=1}^{\infty} \mathbf{E} \left[ \left( \sum_{1 \leq k_1, \dots, k_j \leq n} \|I_{k_1}\| \cdots \|I_{k_j}\| \right)^p 1_{N_s=n} \right] \\ &\leq \left( \frac{(d-j)! \kappa_{d-j}}{d!} \right)^p \sum_{n=1}^{\infty} n^{j \max(p,1)} \mathbf{E}[(\|I_1\| \cdots \|I_j\|)^p 1_{N_s=n}]. \end{aligned}$$

Since the lengths of the segments  $I_1, \dots, I_j$  are independent,  $\mathbf{E}(\|I_1\| \cdots \|I_j\|)^{pr} < \infty$ , where  $r > 1$  satisfies  $pr < \beta_\nu$ . Hence, Hölder's inequality yields that

$$\mathbf{E}[\tilde{V}_j^p 1_{N_s \geq 1}] \leq \left( \frac{(d-j)! \kappa_{d-j}}{d!} \right)^p (\mathbf{E}(\|I_1\| \cdots \|I_j\|)^{pr})^{1/r} \sum_{n=1}^{\infty} n^{j \max(p,1)} \mathbf{P}\{N_s = n\}^{1/q},$$

where  $1/r + 1/q = 1$ . Since all moments of  $N_s$  are finite, the series converges.  $\square$

**Corollary 1.2.** *If  $X$  is the Brownian motion, then  $\mathbf{E}V_j(Z_s)^p < \infty$  for all  $p \geq 0$ , all  $j = 0, \dots, d$  and all  $s \geq 0$ .*

An analogue of Theorem 1.1 holds for random walks. Let  $\{\xi_n, n \geq 1\}$  be a sequence of i.i.d. random vectors in  $\mathbb{R}^d$  and let  $S_n = \xi_1 + \cdots + \xi_n$ ,  $n \geq 1$ . Denote by  $C_n$  the convex hull of the origin and  $S_1, \dots, S_n$ . The following result can be proved similarly to Theorem 1.1.

**Theorem 1.3.** *If  $\mathbf{E}\|\xi_1\|^p < \infty$ , then  $\mathbf{E}V_j(C_n)^p < \infty$  for all  $j = 0, \dots, d$ .*

## 2 Expected intrinsic volumes

Let  $X(t)$ ,  $t \geq 0$ , be a symmetric  $\alpha$ -stable Lévy process in  $\mathbb{R}^d$ . In the Gaussian case all moments of  $V_j(Z)$ ,  $j = 0, \dots, d$ , exist. If  $\alpha < 2$ , then its Lévy measure is  $\nu(dx) = c\|x\|^{-d-\alpha}$  for a constant  $c > 0$ , so that  $\mathbf{E}V_j(Z)^p < \infty$  for each  $p \in [0, \alpha)$ . In this section we calculate the expected intrinsic volumes of  $Z$  assuming that  $\alpha > 1$ .

Recall that the characteristic function of  $X(t)$  can be represented as

$$\mathbf{E} \exp\{i\langle X(t), u \rangle\} = \exp\{-th(K, u)^\alpha\}, \quad u \in \mathbb{R}^d, t \geq 0, \quad (2.1)$$

where  $\langle X(t), u \rangle$  is the scalar product and

$$h(K, u) = \sup\{\langle x, u \rangle : x \in K\}, \quad u \in \mathbb{R}^d,$$

is the support function of a convex body  $K$  called the *associated zonoid* of  $X(1)$ , see [6, 11]. Zonoids are convex bodies that are obtained as limits (with respect to the Hausdorff metric) of zonotopes, i.e. Minkowski sums of segments.

Recall that a random compact set  $Y$  in  $\mathbb{R}^d$  is said to be integrably bounded if  $\|Y\| = \sup\{\|u\| : u \in Y\}$  is integrable. Its *selection expectation*  $\mathbf{E}Y$  is defined as the convex body with support function  $\mathbf{E}h(Y, u)$ ,  $u \in \mathbb{R}^d$ , see [10, Sec. 2.1]. Each zonoid  $K$  can be obtained as the selection expectation of the segment  $[0, \xi]$  with a suitably chosen  $\xi$ .

We start by proving an auxiliary result on the expected  $j$ -dimensional volume of a parallelepiped spanned by random vectors  $\xi_1, \dots, \xi_j \in \mathbb{R}^d$ .

**Theorem 2.1.** *Let  $j \in \{1, \dots, d\}$ . If  $\xi_1, \dots, \xi_j \in \mathbb{R}^d$  are independent integrable random vectors, then*

$$V(\mathbf{E}[0, \xi_1], \dots, \mathbf{E}[0, \xi_j], B^d[d-j]) = \frac{(d-j)!}{d!} \kappa_{d-j} \mathbf{E}D_j(\xi_1, \dots, \xi_j).$$

*Proof.* For all  $k = 1, \dots, j$ , and  $u \in \mathbb{R}^d$ ,

$$h(\mathbf{E}[-\xi_k, \xi_k], u) = \mathbf{E}|\langle \xi_k, u \rangle| = \mathbf{E}[|\langle \xi_k / \|\xi_k\|, u \rangle| \|\xi_k\| \mathbf{1}_{\|\xi_k\| \neq 0}].$$

We define the measure  $\rho_k$  on Borel sets  $A$  in the unit sphere  $\mathbb{S}^{d-1}$  by letting

$$\rho_k(A) = \mathbf{E}[\mathbf{1}_{\xi_k / \|\xi_k\| \in A} \|\xi_k\|].$$

Then

$$h(\mathbf{E}[-\xi_k, \xi_k], u) = \int_{\mathbb{S}^{d-1}} |\langle v, u \rangle| d\rho_k(v), \quad u \in \mathbb{R}^d,$$

meaning that  $\rho_k$  is the generating measure of the zonoid  $\mathbf{E}[-\xi_k, \xi_k]$ , see [15, Th. 3.5.3]. Note that

$$V(\mathbf{E}[0, \xi_1], \dots, \mathbf{E}[0, \xi_j], B^d[d-j]) = 2^{-j} V(\mathbf{E}[-\xi_1, \xi_1], \dots, \mathbf{E}[-\xi_j, \xi_j], B^d[d-j]).$$

By [15, Th. 5.3.2],

$$\begin{aligned} V(\mathbf{E}[0, \xi_1], \dots, \mathbf{E}[0, \xi_j], B^d[d-j]) \\ = \frac{(d-j)!}{d!} \kappa_{d-j} \int_{\mathbb{S}^{d-1}} \cdots \int_{\mathbb{S}^{d-1}} D_j(u_1, \dots, u_j) d\rho_1(u_1) \cdots d\rho_j(u_j). \end{aligned}$$

It remains to identify the integral as  $\mathbf{E}D_j(\xi_1, \dots, \xi_j)$ . □

Let  $M_j$  be a  $d \times j$  matrix composed of  $j$  columns being i.i.d. copies of a random vector  $\xi \in \mathbb{R}^d$ .

**Corollary 2.2.** *If  $\xi \in \mathbb{R}^d$  is an integrable random vector, then*

$$V_j(\mathbf{E}[0, \xi]) = \frac{1}{j!} \mathbf{E} \sqrt{\det M_j^T M_j}, \quad j = 1, \dots, d.$$

*Proof.* In view of (0.2),

$$V_j(\mathbf{E}[0, \xi]) = \frac{1}{j!} \mathbf{E} D_j(\xi_1, \dots, \xi_j), \quad j = 1, \dots, d,$$

where  $\xi_1, \dots, \xi_j$  are i.i.d. copies of the random vector  $\xi \in \mathbb{R}^d$ . It remains to note that

$$\mathbf{E} D_j(\xi_1, \dots, \xi_j) = \mathbf{E} \sqrt{\det M_j^T M_j}. \quad \square$$

**Theorem 2.3.** Let  $X(t)$ ,  $t \geq 0$ , be a symmetric  $\alpha$ -stable Lévy process in  $\mathbb{R}^d$  with  $\alpha > 1$ . Then, for all  $j = 1, \dots, d$ ,

$$\mathbf{E} V_j(Z) = \frac{\Gamma(1 - 1/\alpha)^j \Gamma(1/\alpha)^j}{\pi^j \Gamma(j/\alpha + 1)} V_j(K),$$

where  $K$  is the associated zonoid of  $X(1)$ .

*Proof.* The main idea is to approximate the Lévy process with the random walk  $S_i = X(i/n)$ ,  $i = 0, \dots, n$ . Denote by  $C_n$  the convex hull of  $S_0, S_1, \dots, S_n$ . It is shown in [17] that

$$\mathbf{E} V_j(C_n) = \frac{1}{j!} \sum_{\substack{i_1 + \dots + i_j \leq n \\ i_1, \dots, i_j \geq 1}} \frac{1}{i_1 \dots i_j} \mathbf{E} \sqrt{\left| \det \left( \langle S_{i_m}^{(m)}, S_{i_l}^{(l)} \rangle \right)_{m,l=1}^j \right|}, \quad j = 1, \dots, d,$$

where  $S_n^{(1)}, \dots, S_n^{(d)}$  are i.i.d. random walks that arise from i.i.d. copies  $X^{(1)}, \dots, X^{(d)}$  of the Lévy process. The determinant is known as the Gram determinant and is always non-negative, so that the absolute value can be omitted.

Since  $X(t)$  coincides in distribution with  $t^{1/\alpha} X(1)$  for any  $t > 0$ ,

$$\begin{aligned} \mathbf{E} \sqrt{\det \left( \langle S_{i_m}^{(m)}, S_{i_l}^{(l)} \rangle \right)_{m,l=1}^j} &= \mathbf{E} \sqrt{\det \left( \langle X^{(m)}(i_m/n), X^{(l)}(i_l/n) \rangle \right)_{m,l=1}^j} \\ &= \mathbf{E} \sqrt{\det \left( \langle (i_m/n)^{1/\alpha} X^{(m)}(1), (i_l/n)^{1/\alpha} X^{(l)}(1) \rangle \right)_{m,l=1}^j} \\ &= \frac{(i_1 \dots i_j)^{1/\alpha}}{n^{j/\alpha}} \mathbf{E} \sqrt{\det \left( \langle X^{(m)}(1), X^{(l)}(1) \rangle \right)_{m,l=1}^j} \\ &= \frac{(i_1 \dots i_j)^{1/\alpha}}{n^{j/\alpha}} j! V_j(\mathbf{E}[0, X(1)]), \end{aligned}$$

where the last equality follows from Corollary 2.2. It is shown in [11] that

$$\mathbf{E}[0, X(1)] = \frac{1}{\pi} \Gamma \left( 1 - \frac{1}{\alpha} \right) K,$$

where  $K$  is the associated zonoid of  $X(1)$ . Thus,

$$\mathbf{E} V_j(C_n) = n^{-j/\alpha} V_j \left( \frac{1}{\pi} \Gamma \left( 1 - \frac{1}{\alpha} \right) K \right) \sum_{\substack{i_1 + \dots + i_j \leq n \\ i_1, \dots, i_j \geq 1}} (i_1 \dots i_j)^{1/\alpha - 1} \rightarrow \mathbf{E} V_j(Z) \quad \text{as } n \rightarrow \infty.$$

It follows from the Stolz-Cesàro theorem [12, Th. 1.22] that

$$\begin{aligned} &\lim_{n \rightarrow \infty} n^{-j/\alpha} \sum_{\substack{i_1 + \dots + i_j \leq n \\ i_1, \dots, i_j \geq 1}} (i_1 \dots i_j)^{1/\alpha - 1} \\ &= ((n+1)^{j/\alpha} - n^{j/\alpha})^{-1} \left( \sum_{\substack{i_1 + \dots + i_j \leq n+1 \\ i_1, \dots, i_j \geq 1}} (i_1 \dots i_j)^{1/\alpha - 1} - \sum_{\substack{i_1 + \dots + i_j \leq n \\ i_1, \dots, i_j \geq 1}} (i_1 \dots i_j)^{1/\alpha - 1} \right). \end{aligned}$$

Since

$$\lim_{n \rightarrow \infty} \frac{(n+1)^x - n^x}{xn^{x-1}} = 1$$

for all  $x > 0$ , it remains to find the limit of

$$\begin{aligned} \frac{\alpha}{j} n^{-j/\alpha+1} \sum_{\substack{i_1+\dots+i_j=n+1 \\ i_1, \dots, i_j \geq 1}} (i_1 \dots i_j)^{1/\alpha-1} &= \frac{\alpha}{j} n^{-j/\alpha+1} \sum_{\substack{i_1+\dots+i_j=n+1 \\ i_1, \dots, i_j \geq 1}} \left( \frac{i_1 \dots i_j}{n^j} \right)^{1/\alpha-1} n^{j/\alpha-j} \left( \frac{n}{n} \right)^{j-1} \\ &= \frac{\alpha}{j} \sum_{\substack{i_1+\dots+i_j=n+1 \\ i_1, \dots, i_j \geq 1}} \left( \frac{i_1 \dots i_j}{n^j} \right)^{1/\alpha-1} \left( \frac{1}{n} \right)^{j-1}. \end{aligned}$$

As  $n \rightarrow \infty$ , the limit can be written using the multinomial Beta function related to the Dirichlet distribution, see [4, p. 11], as

$$\frac{\alpha}{j} \int_{\substack{t_1+\dots+t_j=1 \\ t_1, \dots, t_j \geq 0}} (t_1 \dots t_j)^{1/\alpha-1} dt_1 \dots dt_j = \frac{\alpha \Gamma(1/\alpha)^j}{j \Gamma(j/\alpha)} = \frac{\Gamma(1/\alpha)^j}{\Gamma(j/\alpha + 1)}.$$

Finally,

$$\mathbf{E}V_j(Z) = \frac{\Gamma(1/\alpha)^j}{\Gamma(j/\alpha + 1)} V_j \left( \frac{1}{\pi} \Gamma \left( 1 - \frac{1}{\alpha} \right) K \right)$$

and the result follows from the homogeneity of the intrinsic volumes.  $\square$

**Remark 2.4.** By the self-similarity property,  $Z_s$  coincides in distribution with  $s^{1/\alpha}Z$  for  $s > 0$ , whence  $\mathbf{E}V_j(Z_s) = s^{j/\alpha} \mathbf{E}V_j(Z)$ .

**Example 2.5.** If  $X$  is the standard Brownian motion, then  $\alpha = 2$  and  $K = \frac{1}{\sqrt{2}}B^d$ , so that we recover the result of [2]

$$\mathbf{E}V_j(Z) = \binom{d}{j} \left( \frac{\pi}{2} \right)^{j/2} \frac{\Gamma((d-j)/2 + 1)}{\Gamma(j/2 + 1) \Gamma(d/2 + 1)}, \quad j = 1, \dots, d.$$

**Example 2.6.** If  $X(1)$  is spherically symmetric, then

$$\mathbf{E} \exp\{\iota \langle X(1), u \rangle\} = \exp\{-c\|u\|^\alpha\}$$

for  $c > 0$ , see [14, Th. 14.14]. Then  $K = c^{1/\alpha}B^d$ , so that

$$\mathbf{E}V_j(Z) = \binom{d}{j} \frac{\kappa_d}{\kappa_{d-j}} \frac{\Gamma(1 - 1/\alpha)^j \Gamma(1/\alpha)^j}{\pi^j \Gamma(j/\alpha + 1)} c^{j/\alpha}, \quad j = 1, \dots, d.$$

### 3 Interior of the convex hull

It is well known that the convex hull of the Brownian motion in  $\mathbb{R}^d$  contains the origin as interior point with probability 1 for each  $s > 0$ , see [3]. We extend this result for symmetric Lévy processes.

**Theorem 3.1.** *Let  $X(t)$ ,  $t \geq 0$ , be a symmetric Lévy process in  $\mathbb{R}^d$ , such that  $\langle X(1), u \rangle$  has a non-atomic distribution for each  $u \neq 0$ . Then  $\mathbf{P}\{0 \in \text{Int } Z_s\} = 1$  and  $\mathbf{P}\{X(s) \in \text{Int } Z_s\} = 1$  for each  $s > 0$ .*

*Proof.* Note that

$$\mathbf{P}\{0 \in \text{Int } Z_s\} = 1 - \mathbf{P}\{0 \in \partial Z_s\},$$

where  $\partial Z_s$  is the topological boundary of  $Z_s$ . We approximate  $Z_s$  with the convex hull  $C_n = \text{conv}\{X(is/n), i = 0, \dots, n\}$  of the random walk embedded in the process. Observe

that the sequence of events  $\{0 \in \partial C_n\}$  is decreasing to  $\{0 \in \partial Z_s\}$ . Denote by  $Y_n$  the number of faces of  $C_n$  that contain the origin as a vertex. Since  $X(t)$  is symmetric and, in view of the imposed condition,  $\mathbf{P}\{X(t) \in H\} = 0$  for any affine hyperplane  $H$  in  $\mathbb{R}^d$  and  $t > 0$ , [17, Eq. (14),(15)] yield that

$$\mathbf{E}Y_n = 2 \sum_{1 \leq i_2 < \dots < i_d \leq n} \frac{(2n - 2i_d - 1)!!}{i_2(2n - 2i_d)!!} \prod_{k=2}^{d-1} \frac{1}{i_{k+1} - i_k} \sim \frac{2(\log n)^{d-1}}{\sqrt{\pi n}} \quad \text{as } n \rightarrow \infty.$$

Thus,  $\mathbf{E}Y_n \rightarrow 0$  as  $n \rightarrow \infty$ , so that

$$\mathbf{P}\{0 \in \partial Z_s\} = \lim_{n \rightarrow \infty} \mathbf{P}\{0 \in \partial C_n\} = 0.$$

Now we prove the second statement and make the convention that  $X(0-) = 0$ . The time reversal of  $X(t)$ ,  $t \in [0, s]$ , is defined as

$$\tilde{X}(t) = X((s - t)-) - X(s), \quad t \in [0, s].$$

It is well known that  $\tilde{X}(t)$  coincides in distribution with  $-X(t)$ , see [1, Sec. II.1]. By symmetry of the process and the fact that the origin almost surely belongs to the interior of the convex hull,

$$0 \in \text{Int conv}\{X((s - t)-) - X(s), t \in [0, s]\} \quad \text{a.s.}$$

This relation is equivalent to

$$X(s) \in \text{Int conv}\{X((s - t)-), t \in [0, s]\} \subseteq \text{Int } Z_s \quad \text{a.s.} \quad \square$$

The distribution of  $\langle X(1), u \rangle$  is non-atomic for each  $u \neq 0$  if projections of the Lévy measure  $\nu$  on any one-dimensional linear subspace of  $\mathbb{R}^d$  is infinite outside the origin, see [14, Th. 27.4], or if  $X(1)$  has a non-trivial full-dimensional Gaussian component.

#### 4 $L_p$ -geometry of the convex hull

For convex bodies  $L$  and  $M$  that both contain the origin and  $p \in [1, \infty)$ , the  $L_p$ -sum  $L +_p M$  is defined via its support function as

$$h(L +_p M, u)^p = h(L, u)^p + h(M, u)^p, \quad u \in \mathbb{R}^d.$$

If  $p = 1$ , one recovers the Minkowski sum.

Based on (0.1), the mixed volume  $V(L[d - 1], M[1])$  can be defined as

$$V(L[d - 1], M[1]) = \frac{1}{d} \lim_{t \downarrow 0} \frac{V_d(L + tM) - V_d(L)}{t}.$$

The  $L_p$ -generalisation of this mixed volume is defined by

$$V_p(L, M) = \frac{p}{d} \lim_{t \downarrow 0} \frac{V_d(L +_p t^{1/p} M) - V_d(L)}{t},$$

where it is required that  $L$  and  $M$  are convex bodies with the origin as interior point, see [15, Eq. (9.11)].

We generalise the expression for  $\mathbf{E}V(L[d - 1], Z[1])$  from [6] for the  $L_p$ -case and symmetric  $\alpha$ -stable Lévy processes in  $\mathbb{R}^d$  with  $\alpha \in (1, 2]$ . The case  $\alpha = 2$  is considered separately, since the integrability condition is different and in this case we obtain an explicit expression.

**Theorem 4.1.** Let  $L$  be a convex body in  $\mathbb{R}^d$  with the origin as an interior point and  $X(t)$ ,  $t \geq 0$ , be a symmetric  $\alpha$ -stable Lévy process in  $\mathbb{R}^d$  with  $\alpha \in (1, 2)$ , such that  $\langle X(t), u \rangle$  is nondegenerate for all  $u \neq 0$ . Then, for all  $p \in [1, \alpha)$ ,

$$\mathbf{E}V_p(L, Z) = \mathbf{E}\left(\sup_{0 \leq t \leq 1} R(t)\right)^p V_p(L, K),$$

where  $R(t)$  is a symmetric  $\alpha$ -stable Lévy process in  $\mathbb{R}$  with the same parameter  $\alpha$  as  $X(t)$  and scale parameter 1, and  $K$  is the associated zonoid of  $X(1)$ .

*Proof.* It follows from the definition of  $\alpha$ -stability that the one-dimensional Lévy process  $\langle X(t), u \rangle$  is symmetric  $\alpha$ -stable. In view of (2.1),

$$\mathbf{E} \exp\{is\langle X(t), u \rangle\} = \exp\{-t|s|^\alpha h(K, u)^\alpha\}.$$

The finite-dimensional distributions of the process  $\langle X(t), u \rangle$ ,  $t \geq 0$ , coincide with those of the process  $h(K, u)R(t)$ ,  $t \geq 0$ , and

$$\begin{aligned} \mathbf{E}h(Z, u)^p &= \mathbf{E}(\sup\{\langle X(t), u \rangle : t \in [0, 1]\})^p \\ &= h(K, u)^p \mathbf{E}\left(\sup_{0 \leq t \leq 1} R(t)\right)^p. \end{aligned}$$

The last expectation is finite if  $p < \alpha$ .

It is shown [15, Eq. (9.18)] that

$$V_p(L, M) = \frac{1}{d} \int_{\mathbb{S}^{d-1}} h(M, u)^p S_{p,0}(L, du),$$

where  $S_{p,0}(L, \cdot)$  is a measure on the unit sphere. It follows from Fubini's theorem that

$$\begin{aligned} \mathbf{E}V_p(L, Z) &= \mathbf{E}\left(\frac{1}{d} \int_{\mathbb{S}^{d-1}} h(Z, u)^p S_{p,0}(L, du)\right) \\ &= \frac{1}{d} \int_{\mathbb{S}^{d-1}} \mathbf{E}h(Z, u)^p S_{p,0}(L, du) \\ &= \mathbf{E}\left(\sup_{0 \leq t \leq 1} R(t)\right)^p \frac{1}{d} \int_{\mathbb{S}^{d-1}} h(K, u)^p S_{p,0}(L, du) \\ &= \mathbf{E}\left(\sup_{0 \leq t \leq 1} R(t)\right)^p V_p(L, K). \end{aligned} \quad \square$$

**Theorem 4.2.** Let  $L$  be a convex body in  $\mathbb{R}^d$  with the origin as an interior point and  $X(t)$ ,  $t \geq 0$ , be the standard Brownian Motion in  $\mathbb{R}^d$ . Then, for all  $p \in [1, \infty)$ ,

$$\mathbf{E}V_p(L, Z) = \frac{2^{p/2}}{\sqrt{\pi}} \Gamma\left(\frac{p+1}{2}\right) V_p(L, B^d).$$

*Proof.* It is obvious that the finite-dimensional distributions of the process  $\langle X(t), u \rangle$ ,  $t \geq 0$ , coincide with those of the process  $h(K, u)R(t)$ , where  $R(t) = \sqrt{2}W(t)$  for the Brownian motion  $W$  in  $\mathbb{R}$ . Then, for all  $p \geq 1$ ,

$$\mathbf{E}\left(\sup_{0 \leq t \leq 1} R(t)\right)^p = \mathbf{E}\left(\sup_{0 \leq t \leq 1} \sqrt{2}W(t)\right)^p = \frac{2^p}{\sqrt{\pi}} \Gamma\left(\frac{p+1}{2}\right),$$

so that the result follows by repeating the arguments from the proof of Theorem 4.1 and noticing that the associated zonoid of  $X$  is  $K = B^d/\sqrt{2}$ .  $\square$



## 5 Limit theorems for scaled convex hull

Recall that  $T_1$  denotes the first exit time of the process  $X$  from the unit ball. It is obvious that  $\mathbf{E}T_1 \leq \mathbf{E} \min\{j \geq 1 : \|X(j)\| > 1\} < \infty$ , see [13, Th. 1].

The standard (Skorohod) metric on the space  $D([0, 1], \mathbb{R}^d)$  of all right-continuous  $\mathbb{R}^d$ -valued functions with left limits defined on  $[0, 1]$  is defined by

$$d_{J_1}(f(t), g(t)) = \inf_{\lambda \in \Lambda} \{\|f(\lambda(t)) - g(t)\|_\infty + \|\lambda(t) - t\|_\infty\},$$

where  $\Lambda$  is the set of strictly increasing functions  $\lambda$  mapping  $[0, 1]$  onto itself, such that both  $\lambda$  and its inverse  $\lambda^{-1}$  are continuous.

**Lemma 5.1.** *The map that associates with a function  $f \in D([0, 1], \mathbb{R}^d)$  the closed convex hull of its range  $\{f(t) : 0 \leq t \leq 1\}$  is continuous in the Hausdorff metric on  $\mathcal{K}_d$ .*

*Proof.* It suffices to note that the Hausdorff distance between the closed ranges of  $f, g \in D([0, 1], \mathbb{R}^d)$  is dominated by  $d_{J_1}(f, g)$ , and the Hausdorff distance between the convex hulls of compact sets equals the Hausdorff distance between the sets themselves.  $\square$

**Theorem 5.2.** *Assume that  $X(t), t \geq 0$ , is a Lévy process in  $\mathbb{R}^d$  such that  $X(T_1)$  lies in the domain of attraction of a strictly  $\alpha$ -stable random vector  $\eta$ , that is the sum of  $n$  i.i.d. copies of  $X(T_1)$  scaled by  $(n^{1/\alpha}\ell(n))^{-1}$  with a slowly varying function  $\ell$  converges in distribution to  $\eta$ . Then  $(t^{1/\alpha}\ell(t))^{-1}Z_t$  converges in distribution to  $\text{conv}\{Y(s) : 0 \leq s \leq (\mathbf{E}T_1)^{-1}\}$ , where  $Y$  is a Lévy process such that  $Y(1)$  coincides in distribution with  $\eta$ .*

*Proof.* Without loss of generality assume that  $\ell(n) = 1$  for all  $n$ . As in the proof of Theorem 1.1, split the path of the Lévy process into several parts using the stopping times defined in (1.1) and consider the renewal process  $N_s$ . Then  $S_n = X(T_n)$ ,  $n \geq 1$ , forms a random walk embedded in the process and

$$\text{conv}\{0, S_1, \dots, S_{N_t}\} \subseteq Z_t \subseteq \text{conv}\{0, S_1, \dots, S_{N_t}\} + B^d$$

for all  $t > 0$ , so that

$$\frac{\text{conv}\{0, S_1, \dots, S_{N_t}\}}{N_t^{1/\alpha}} \left(\frac{N_t}{t}\right)^{1/\alpha} \subseteq \frac{Z_t}{t^{1/\alpha}} \subseteq \frac{\text{conv}\{0, S_1, \dots, S_{N_t}\}}{N_t^{1/\alpha}} \left(\frac{N_t}{t}\right)^{1/\alpha} + \frac{B^d}{t^{1/\alpha}}.$$

In the square integrable case, the convergence of  $n^{-1/2} \text{conv}\{0, S_1, \dots, S_n\}$  to the convex hull of the Brownian motion is established in [18, Th. 2.5]. In the general case, Donsker's invariance principle with stable limits [19, Th. 4.5.3] implies that  $n^{-1/\alpha} S_{\lfloor nt \rfloor}$ ,  $0 \leq t \leq 1$ , converges weakly to the process  $Y(t)$ ,  $0 \leq t \leq 1$ , in  $(D, d_{J_1})$ . Using the single probability space argument, the convergence holds if  $n$  is replaced by  $N_t$  and  $t \rightarrow \infty$ . Lemma 5.1 and the continuous mapping theorem yield the weak convergence of  $N_t^{-1/\alpha} \text{conv}\{0, S_1, \dots, S_{N_t}\}$  to  $\text{conv}\{Y(t) : 0 \leq t \leq 1\}$  in  $\mathcal{K}_d$  with the Hausdorff metric.

The final statement follows from  $N_t/t \rightarrow (\mathbf{E}T_1)^{-1}$  a.s. as  $t \rightarrow \infty$ , taking into account the scaling property of the process  $Y$  and the fact that  $t^{-1/\alpha} B^d \rightarrow \{0\}$  as  $t \rightarrow \infty$ .  $\square$

Recall that  $\nu$  denotes the Lévy measure of  $X$ .

**Theorem 5.3.** (i) *If*

$$\int_{\|x\|>1} \|x\|^2 \nu(dx) < \infty, \quad (5.1)$$

*then  $X(T_1)$  belongs to the domain of attraction of the Gaussian law.*

- (ii) Assume that (5.1) does not hold,  $\nu(\{x : \|x\| \geq r\})$ ,  $r > 0$ , is a regularly varying function at infinity with exponent  $-\alpha$  for  $\alpha \in (0, 2)$ , and

$$\frac{\nu(\{x : \|x\| \geq r, x/\|x\| \in A\})}{\nu(\{x : \|x\| \geq r\})} \rightarrow \frac{\Lambda(A)}{\Lambda(\mathbb{S}^{d-1})} \quad \text{as } r \rightarrow \infty$$

for all Borel subsets  $A$  of the unit sphere  $\mathbb{S}^{d-1}$  such that  $\Lambda(\partial A) = 0$ , where  $\Lambda$  is a finite Borel measure on the unit sphere which is not supported by any proper linear subspace. Then  $X(T_1)$  belongs to the domain of attraction of an  $\alpha$ -stable law.

*Proof.* It is well known that a Lévy process  $X$  can be expressed as the sum of three independent Lévy processes  $X_1, X_2, X_3$ , where  $X_1$  is a linear transform of a Brownian motion with drift,  $X_2$  is a compound Poisson process having only jumps of norm strictly larger than 2 and  $X_3$  is a pure jump process having jumps of size at most 2. This decomposition is known as the Lévy-Itô decomposition. It is obvious that

$$X(T_1) = X(T_1-) + X_2(T_1) - X_2(T_1-) + X_3(T_1) - X_3(T_1-).$$

Since the norm of  $X(T_1-) + X_3(T_1) - X_3(T_1-)$  is at most 3, it is in the domain of attraction of the normal distribution.

The compound Poisson process

$$X_2(t) = \sum_{i: \tau_i \leq t} \xi_i$$

is given by the sum of i.i.d. random vectors  $\xi_i$ ,  $i \geq 1$ , with the distribution given by  $\nu$  restricted onto the complement of the ball of radius 2 and normalised to become a probability measure. Here  $\{\tau_i, i \geq 1\}$  is a homogeneous Poisson process on  $\mathbb{R}_+$ . Thus,  $T_1 \leq \tau_1$  a.s., and  $X_2(T_1) - X_2(T_1-)$  is zero if  $T_1 < \tau_1$ , while otherwise it equals  $\xi_1$ . If (5.1) holds, then  $\mathbb{E}\|\xi_1\|^2 < \infty$ , so that  $X_2(T_1) - X_2(T_1-)$  belongs to the domain of attraction of a normal law. Otherwise, the conditions of the theorem guarantee that  $X_2(T_1) - X_2(T_1-)$  belongs to the domain of attraction of an  $\alpha$ -stable law, see [9, Th. 8.2.18]. Then  $X(T_1)$  also belongs to the domain of attraction of the same law.  $\square$

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